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THE ANALYTICAL TREATMENT OF NEWTON'S PROBLEMS.

IT is a great help to the proper understanding of the meaning of Newton's propositions to consider the formulation of these propositions in analytical language. In this article, after a mention is made of Leibniz's analytical work of 1689 (§ I), Varignon's of 1690 (§ II), Maclaurin's of 1742 (§ III), and of some work of Johann Bernoulli, Keill, Hermann, and others near the beginning of the eighteenth century (§ IV), the contributions of Lagrange (1788) and the integrals of the dynamical equations are considered in greater detail (§§ V, VI). In connection with these integrals, the problems of the first three and then two bodies are treated in the most general way possible (§ VII), and, in § VIII, a simple case of integration noticed by Jacobi is given and some further simplifications added. In § IX a short analytical summary of much of Newton's work on the dynamics of a particle is given. In § X the work of Möbius, Hamilton, and Grassmann on vector methods, which is of such great importance to analytical mechanics, is indicated, and the logical problem of the principles of mechanics and the light thrown on it by this work are discussed.

I.

The first memoirs that we shall have to consider are two by Leibniz. The *Acta Eruditorum* of Leipsic gave in

1688 a very long review of Newton's *Principia,* and in 1689 appeared in these *Acta* two papers by Leibniz entitled "Schediasma de Resistentia medii et Motu projectorum gravium in medio resistente"[1] and "Tentamen de Motuum Caelestium causis,"[2] the latter of which may be described[3] as an attempt to combine Cartesian and Newtonian ideas. Leibniz stated[4] that he had not seen the *Principia* when he wrote these papers, but that he had seen the above review. Not being able to understand by what methods the most important of Newton's discoveries as described in that review, viz., the elliptical motion of the heavenly bodies, could have been proved mathematically by Newton from the data obtained by observation and law of attraction governing that motion, Leibniz gave this proposition as deduced by himself with the help only of his infinitesimal calculus. Leibniz obtained[5] the equation of the centripetal force, which is certainly in accordance with Newton's sixth Proposition of his first Book but cannot be said[6] to "coincide with it"; for Leibniz found, in a somewhat different notation, the now well-known equation for the force acting along the radius vector.

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[1] *Leibnizens mathematische Schriften* (edited by C. I. Gerhardt), Abt. II (which is Vol. VI of the 3d series of G. H. Pertz's edition of *Leibnizens gesammelte Werke*), Halle, 1860, pp. 135-143.

[2] *Ibid.,* pp. 144-161. Another manuscript of Leibniz and a letter to Huygens, both relating to this paper, are printed in *ibid.,* pp. 161-193. Cf. also Gerhardt's remarks, *ibid.,* pp. 10-12.

[3] Cf. Rosenberger, *Isaac Newton und seine physikalischen Principien,* Leipsic, 1895, pp. 231-232.

[4] According to Brougham and Routh (*Analytical View of Sir Isaac Newtons' "Principia,"* London, 1855, pp. 438-439). However, in Gerhardt's publication, in neither of these two papers does Leibniz mention the fact that he himself was at Rome when the papers were written, and only in the second Newton is mentioned. On the subject of the law of gravitational attraction Leibniz says (Gerhardt's ed., p. 157): "Video hanc propositionem jam tum innotuisse etiam viro celeberrimo Isaaco Newtono, ut ex relatione *Actorum* apparet, licet inde non possim judicare, quomodo ad eam pervenerit."

[5] Gerhardt's edition, pp. 156-157; cf. pp. 176-178.

[6] Brougham and Routh, *op. cit.,* p. 441.

II.

In 1690, Varignon published a paper in which he translated mechanical conceptions into terms of Leibniz's calculus, writing dx/dt for the velocity (v) and d^2x/dt^2 or dv/dt for the acceleration, and so on. Rosenberger[7] said that this translation was made here for the first time, but this remark, by what we have seen in the first section, would seem to be a mistake.

III.

A systematic use of a fixed origin of coordinates, and indeed rectangular coordinates, x, y, and z, was first made by Maclaurin in his work: *A Treatise of Fluxions,* published at Edinburgh in 1742.[8] The use of fixed directions along which to resolve forces makes it possible to write out the equations of motion much more simply. In fact, we know that, in the case of a stone whirled round at the end of a string for instance, the result of the action of a force may be to change constantly the direction of motion of a body without bringing it any nearer to the attracting center of force. When Huygens and Newton investigated the motion of a particle revolving in a circular orbit round the center, they found—in modern notation—that there was an acceleration along the radius vector r of amount $r(d\vartheta/dt)^2$, although the particle never approaches the center of the circle. The velocity along r is measured by dr/dt, but the *acceleration* along r is not measured by d^2r/dt^2; a term due to the change in the direction of the velocity must be taken into account. We calculate this in the simplest way when we find the velocity and acceleration along r from the observation that the velocities and accelerations along the axes of x and y are dx/dt, dy/dt, and d^2x/dt^2,

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[7] *Op. cit.*, p. 229.

[8] Vol. I, pp. 386-390 (arts. 465-469), and Vol. II, p. 710 (art. 884).

d^2y/dt^2, respectively, and that:

$$x = r\cos\vartheta,\ y = r\sin\vartheta.$$

We then find that the acceleration along r is

$$(d^2x/dt^2)\cos\vartheta + (d^2y/dt^2)\sin\vartheta = d^2r/dt^2 - r(d\vartheta/dt)^2,$$

while the acceleration perpendicular to r is

$$(d^2y/dt^2)\cos\vartheta - (d^2x/dt^2)\sin\vartheta = (1/r)d/dt(r^2\, d\vartheta/dt).$$

It is to be observed that Maclaurin was the first to use Cartesian coordinates systematically in mechanical problems, and that Euler in 1736 and D'Alembert in 1743 used what are called "intrinsic" or "natural" coordinates.[9] In particular, Euler seems to have been the first to give explicitly the now well-known formulas expressing the tangential and normal accelerations of a particle describing a curve.

IV.

Johann Bernoulli (1710), in a letter to Hermann which was published in the *Mémoires* of the French Academy, gave the proposition $f = h^2r/p^3\varrho$, which I consider in another article. It was afterward given, in different wording, as a deduction from the sixth Proposition of the first Book in the second edition (1713) of Newton's *Principia*. Keill[10] gave a demonstration of Bernoulli's theorem which was much more roundabout, and as of a theorem which Demoivre had communicated to him, adding that Demoivre also informed him that Newton had invented a similar matter before.[11]

In the same paper Bernoulli maintained that Newton had assumed without demonstration that no other curves than conics can be the trajectories of bodies moving under the influence of a centripetal force varying inversely as the square of the distance. It certainly cannot be maintained that Newton's phraseology in his seventeenth Proposition is free from ambiguity, but, according to Brougham and Routh,[12] this accusation brought by Bernoulli is not correct. Newton "certainly gives a very concise and compendious demonstration, but he states distinctly that, the focus and point being given, and the tangent given in position, a conic section may be described which shall at that point of contact have a given curvature; that the curvature is given from the velocity and central force being given; and that two orbits touching each other cannot be described with the same centripetal force and velocity. . . . But it must also be observed, as Laplace has remarked, that Newton has in a subsequent problem shown how to find the curve in which a body must move with a given velocity, initial direction, and position; and since when the centripetal force is inversely as the square of the distance the curve is shown to be one or other of the conic sections, he has thus demonstrated the proposition in question; so if he had not done so in the Corollary to one problem, he has in the solution of another."[13]

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[9] See the description and references in the *Bulletin of the Amer. Math. Soc.*, Vol. XXII, 1915, pp. 139-141.

[10] *Phil. Trans.*, Vol. XXVI, p. 74.

[11] Brougham and Routh, *op. cit.*, pp. 46-48.

However, in quite modern times, Paul Stäckel[14] has said that "Newton deduced his law of attraction from Kepler's laws. That conversely a central force inversely proportional to the square of the distance always leads to a Kepler's motion in a conic section was then shown by Johann Bernoulli."

In the same paper, Bernoulli objected to a solution of the inverse problem given by Hermann in the same volume of the *Mémoires,* which had been communicated to him before it was presented to the Academy, in which use was made of the expression $-d^2x\sqrt{(x^2+y^2)}/x$ for the centripetal force. Bernoulli's objection was that the manner of working out the problem showed that Hermann was previously aware of the solution in the case of the conic sections. Further, Hermann omitted a constant in his integration, and, by adding it, Bernoulli showed that the equation which Hermann found, when thus corrected, expresses the conic sections generally.[15]

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[12] *Ibid.*, pp. 58, 62.

[13] "*Système du Monde,* liv. V, chap. V. It is to be observed that the seventeenth Proposition of Book I is exactly the same in the first as in the subsequent editions, except for the immaterial addition of a few lines to the demonstration. Consequently, Bernoulli must have been aware of it when he wrote in 1710."

[14] "Elementare Dynamik der Punktsysteme und starren Körper" (1908), *Encykl. der math. Wiss.*, Vol. IV, Part I, 1901-1908, p. 494.

V.

A great change in the treatment of mechanical problems came about as a consequence of the work of Lagrange. With the gradual development of Lagrange's method of dealing with the general problem of dynamics I have dealt in the supplementary volume to Mach's *Mechanics.*[16] Here we must notice that Lagrange, by reducing all the problems of analytical mechanics to the systematic development of one set of equations, almost accomplished the enormous progress of making of mechanics a subject of pure mathematics. The whole question of the motion of mutually attracting particles thus came to be a very small part of Lagrange's inspired routine,[17] which was greatly developed by Poisson, Hamilton, Jacobi, and many others.

In the work of these men the problems of dynamics were reduced to the transformation and integration of the differential equations of mechanics. Of course, there was a reaction against this analytical tendency, and the often contrary tendencies of the work of Poinsot, Thomson and Tait, and Helmholtz give examples of this. Of the vectorial methods associated with the names of Grassmann and Hamilton we will speak later.

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[15] Brougham and Routh, *op. cit.,* pp. 58-59.

[16] Chicago and London, 1915, pp. 90, 96, 98-99.

[17] Cf. *Méchanique analitique,* Paris, 1788, pp. 262-265, 288-295.

Newton was much occupied with the demolition of Descartes's theory of vortices, and consequently paid some attention to the motion of particles in resisting mediums. But when it came to be believed, principally as a result of the great success of the Newtonian treatment of gravitation, that the forces which appear in the important problem of astronomical dynamics act at a distance and not by means of a medium, the question of resistance and friction rather fell into the background, at least with theoretical men of science whose interest was concentrated on problems of celestial dynamics. Thus Lagrange, writing toward the end of the eighteenth century, hardly touched upon the motion of a particle in a resisting medium, and his example was followed by Jacobi and Kirchhoff.[18]

One point that comes out very strongly in Lagrange's reduction of the dynamical problem is the clear separation of the principles of mechanics into those theorems of a general nature that give us integrals of the dynamical equations, and those principles that give us the dynamical equations themselves.[19] Thus, the "principle of *vis viva*" of Huygens, Johann Bernoulli, and Daniel Bernoulli gives us a finite equation between the velocities of the bodies considered and their coordinates in space, so that if there is only one such coordinate the equation obtained from the principle spoken of is sufficient completely to solve the problem. This is the case with the problem of the center of oscillation. In general, the principle mentioned gives a first integral of the mechanical equations. Again, when there is no outer force acting on a system of bodies, we at once get an equation expressing that the coordinates of the center of gravity of the system have their second differential quotients equal to zero. These three differential equations can be immediately integrated and give three first integrals of the system of mechanical equations, expressing that the velocity of the center of gravity is constant. These three equations can be integrated again and give three more integrals of the system of equations. Lastly, we have what is known as "the principle of the conservation of areas," which was discovered in the period 1743 to 1746 by Daniel Bernoulli, Euler, and D'Arcy under somewhat different forms. In D'Arcy's original form the principle appears as a generalization of Newton's theorem on the equable description of areas; and is that the sum of the product of the mass of each body by the area that its radius vector describes round a fixed center is always proportional to the time. In 1749 and 1752 D'Arcy gave his principle in the equivalent form of the constancy of the sum of the products of the masses, the velocities, and the perpendiculars drawn from the center on the direction of the velocities.[20]

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[18] *Encykl. der math. Wiss.,* Vol. IV, Part I, p. 472.

[19] *Méchanique analitique,* pp. 184-188; *Ostwalds Klassiker,* No. 191, pp. 76-77.

Lagrange contrasts with these principles, which give integrals of the system of dynamical equations, the "principle of least action," which gives, at least in the form which Lagrange gave it, a condensed formula for the whole system of these equations.[21] To this principle, we may add Gauss's principle of least constraint and Hamilton's principle.[22]

VI.

Lagrange stated the object of his *Méchanique analitique* of 1788 to be to "reduce the theory of this science and the art of solving its problems to general formulas of which simple development gives all the equations necessary for the solution of each problem." In his book there were no figures: "The methods exposed do not require geometrical or mechanical constructions or reasoning, but merely algebraical operations in a regular and uniform progress." It results from this that new light was thrown upon the various principles of mechanics, for we can only see clearly the nature and limitations of the methods used for the solution of problems of dynamics when we write out in the most general way possible the equations which hold the solutions and then see how the search for integrals is conducted. It then appears that, for example, the theorem of the equable description of areas is a particular case of the fact that a certain combination of our dynamical equations is at once integrable.

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[20] *Ostwalds Klassiker,* No. 191, p. 84.

[21] Cf. Jourdain, *The Principle of Least Action,* Chicago and London, 1913, pp. 75, 78.

[22] These principles, which do not give integrals but transformations of the equations of motion, are discussed in *Ostwalds Klassiker,* No. 167, and Jourdain's *Least Action.*

All the three general theorems that we have mentioned which give integrals, hold good in problems on the motion of n mutually attracting bodies for which resistance of a medium is not to be taken into account. This is the important case of the differential equations of astronomy, and the problem of the "integration of the equations of mechanics" is usually assumed, after Jacobi, to mean the integration of a system of ordinary equations of the second order for which integrals can be found by the above three principles. It may be remarked that the integrals given by the principle of the center of gravity are six in number of which three are "first integrals," that is to say, if φ is a function of the coordinates, the time, and the velocity, and in virtue of the equations $d\varphi/dt = 0$, then the equation $\varphi =$ a constant is a "first integral."[23]

VII.

Let us now consider the problem of three mutually attracting particles in space. We shall then see an easily generalized instance of the differential equations of dynamics for which all the three integrals mentioned hold good, and can then treat the problem of two particles as a particular case. It will clearly be of great advantage to consider the problem quite in general and not introduce any simplifying conditions about the selection of axes.

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[23] On integrals of the equations of mechanics, see *Encykl. der math. Wiss.*, Vol. IV, Part I, pp. 462-469, 474-478.

We will, then, take up the equations of motion in the form given to them by Lagrange, and in which everything depended on differential quotients of half the *vis viva,* or T, and the force function or negative potential energy U, for the case of three mutually attracting particles. In this case we will point out the way to find the integrals, and then point out how Newton's problem of two bodies can be solved by this very general routine method which throws so much light on the particular treatment of dynamical problems.

Each Lagrangian equation is of the form

$$(d/dt)\,(\delta T/\delta q') - \delta T/\delta q = \delta U/\delta q,$$

where q is the general type of a generalized coordinate, q' denotes dq/dt, and the deltas denote partial differentiations. If we have three mutually attracting particles, each of them perfectly free in space, the generalized coordinates are nine in number, and consequently there are nine Lagrange's equations of motion. If the generalized coordinates q are rectangular coordinates, the function T is a homogeneous quadratic function of the terms q' and does not contain any terms q; so that the second term on the left of each of Lagrange's equations vanishes. The problem of finding integrals then reduces to the combination of all the terms with one another and with coefficients in such a way that they form a complete differential with respect to the time.

Thus, if we multiply each equation throughout by the differential quotient with respect to t of its appropriate q, and add all the equations, we get, after some simple mathematical transformations, the result

$$(d/dt)(T - U) = 0,$$

which is at once integrable and is the equation of *vis viva*. Again, it is easily seen that, in the case of gravitation,

$$U = -m_1m_2/r_{1,2} - m_1m_3/r_{1,3} - m_2m_3/r_{2,3},$$

where $r_{1,2}$ is the distance between m_1 and m_2, and so on. Then we get three equations of which one is

$$\delta U/\delta q_1 + \delta U/\delta q_4 + \delta U/\delta q_7 = 0.$$

Here we have, when we substitute for $\delta U/\delta q$, a set of three equations which are immediately integrable twice over. These are the integrals given by what is known as the "principle of the conservation of motion of the center of gravity."

In the third place, we can get, if for simplicity we represent by p with an appropriate suffix the expression $\delta T/\delta q'$ with its suffix, three other equations of which one is

$$(d/dt)\ (q_1p_2 - q_2p_1 + q_4p_5 - q_5p_4 + q_7p_8 - q_8p_7) = 0.$$

These integrals represent what is known as the "principle of the conservation of areas."

Whatever the number of the bodies may be, we always get these seven first integrals of the system; but when we go beyond two bodies these integrals are not enough in number to solve the problem. Except in certain special cases, which unfortunately happen to be of great practical value to astronomers dealing with the question of the motions of the sun, earth, and moon, the problem of three bodies has not yet been solved. Indeed, fairly recently H. Bruns has proved strictly that there are no other algebraic first integrals of the problem of n bodies than those seven mentioned. It must be remembered that the three first integrals of the principle of the center of gravity can be immediately integrated again, so that we have ten integrals in all. The problem of three bodies has been reduced still further, but it is not necessary to speak of that in this place.

It is of great interest to see how Lagrange treated the problems of two and more bodies in his *Méchanique*. He found that it is always possible to refer the bodies to moving axes through any one of the bodies the coordinates of which do not depend merely on the position relative to this body (m) being capable of elimination by means of the integrals of the principle of the center of gravity. Thus the motions of other bodies round m could be determined by differential equations. If we descend to the case of two bodies, we find without difficulty that the same equations hold as for motion of a body round a fixed center of force; and this is one of the great discoveries of Newton to which we refer in another article.[24]

VIII.

The shortest method of integrating the equations of motion of a single particle in the xy-plane under the influence of a central force varying inversely as the square of the distance was given by Jacobi in 1842.[25] We have, since $x = r\cos\vartheta$ and $y = r\sin\vartheta$, and thus $x/r = \cos\vartheta$ and $y/r = \sin\vartheta$, $d^2x/dt^2 = -kx/r^3$, $d^2y/dt^2 = -ky/r^3$, where k is a constant, so that we can at once deduce

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[24] For further information on the subject of the equations of dynamics we may refer to George Boole's *Treatise on Differential Equations*, 2d ed., Cambridge and London, 1865, pp. 312-315; J. Graindorge, *Intégration des équations de la mécanique*, Liège and Bruxelles, 1889; A. Cayley, "Report on Theoretical Dynamics," *Rep. Brit. Ass.*, 1857, pp. 1-42; P. Painlevé, *Leçons sur l'intégration des équations différentiales de la mécanique*, Paris, 1895; and the articles in Vol. II, Parts II and III, 1900, of the *Encykl. der math. Wiss.*, especially pp. 343-346, and the articles on astronomical mechanics in the later volumes of this *Encyklopädie*.

[25] "De motu puncti singularis," *Journ. für Math.*, Vol. XXIV, 1842, pp. 5-27; *Werke*, Vol. IV, pp. 265-288 (especially pp. 281-282). This method was simplified by Thomson and Tait; cf., for example, Tait's article "Mechanics" in the 9th ed. of the *Encyclopædia Britannica* (Vol. XV, 1883, p. 685).

$$y(d^2x/dt^2) - x(d^2y/dt^2) = 0.$$

By integrating the last equation and substituting for x and y their values in terms of r and ϑ, we have

$$r^2 d\vartheta/dt = h,$$

where h is a constant.[26] Substituting $r^2 = h/\vartheta'$ in the first pair of equations and noticing that, for example, $(d/dt)(\sin\vartheta) = \vartheta'\cos\vartheta$, we write the pair in the immediately integrable form:

$$d^2x/dt^2 = -(k/h)(d/dt)(y/r),$$

$$d^2y/dt^2 = (k/h)(d/dt)(x/r).$$

Notice that these equations are only immediately integrable because the law of force is such as to make the coefficient of the differential quotient on the right a constant. Integrating, we get x' and y'; and again using the equation $xy' - yx' = h$, we get the equation between x and y:

$$h - \alpha x + \beta y = kr/h,$$

where α and β are constants, which represents a conic section.

IX.

We may condense in a few lines the essence of much of what Newton did in the dynamics of a particle.[27]

The two equations at the beginning of the preceding section are easily expressed in polar coordinates, and by an obvious combination of them we get the first integral $r^2 d\vartheta/dt = h$. We thus find that this integral, together with the equation $d^2r/dt^2 - r(d\vartheta/dt)^2 = -\mathrm{R}$, where R takes the place of k/r^2 for the sake of greater generality, form an equivalent pair to the pair we started with. The integral is the principle of areas, and by it we can eliminate $d\vartheta/dt$ from the second equation. We thus get $d^2r/dt^2 = h^2/r^3 - R$.

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[26] Here we may remark that this integral of areas is given at once from Lagrange's equation of motion of the particle if we notice that neither T nor U contains θ, though T does contain θ'. In this case, then, Lagrange's equation reduces to $(d/dt)(\delta \mathrm{T}/\delta\theta') = 0$. Here θ is said to be an "ignorable coordinate."

[27] Cf. Gustav Kirchhoff, *Vorlesungen über mathematische Physik*: *Mechanik,* 2d ed., Leipsic, 1877, pp. 7-10.

Now, the equation of a conic is $r = -ex + c$, where e is the eccentricity. From this, by differentiation, and the equations of motion, we find that $d^2r/dt^2 = Rc/r - R$ if a conic is to be described under a central force. But we have found that, under a central force, $d^2r/dt^2 = h^2/r^3 - R$. Comparing these two expressions, we have $Rc = h^2/r^2$; and thus R must be inversely proportional to r^2. Conversely, if R is inversely proportional to r^2, the orbit must be a conic, and this was Newton's method of verification.

X.

The methods of geometrical analysis introduced by Möbius, Hamilton, Grassmann, and others, and known by the names of "barycentric calculus," "*Ausdehnungslehre*," "calculus of quaternions," and so on, has become of great importance in mechanics, principally owing to the conformity with mechanical ideas of the conceptions of "multiplication" in these calculi. Indeed it was Lagrange's *Méchanique* that gave to Grassmann the original ideas of "inner" and "outer" multiplication that characterized his "theory of extension." Inner multiplication has exactly the same interpretation in space as the measure of "work," and outer multiplication is interpretable as the area described by a particle revolving round a center of force in which the direction is taken into account. In the reference of moving bodies to a system of Cartesian or other axes in space, there is, as a rule, no good reason why these particular axes should be chosen rather than others[28] and this often results in very complicated formulas. The vectors which are fundamental in all of the methods of analysis mentioned can be, if we like, decomposed into three mutually rectangular vectors along Cartesian axes, but a great advance in dealing with mechanical problems is then lost, because the parallelogram law according to which vectors, like velocities and forces, are compounded fades away from our sight.

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[28] Of course this does not apply to "intrinsic" coordinates.

The term "geometrical analysis" is misleading in that it seems to imply that geometrical vectors enter into the working of the calculus. But this is not the case: the analysis of vectors does not even require elementary geometrical knowledge of constructions in space.[29] It is quite possible to treat the whole of mechanics in the way of which an example was given by Lagrange. Whether this is desirable from a teacher's point of view does not concern us here: what does concern us here and what I wish to emphasize is the illumination brought to the mind of both student and teacher by the consideration of the particular artifices used by Newton and others for the integration of mechanical equations as special cases of the general theorems relating to the integrals of systems of equations of mechanics. The illumination is somewhat like the illumination given to the student when he contemplates the binomial theorem of his early work on algebra in the light of Taylor's theorem.

From the point of view of inquiry into the principles of mechanics, a combination of the purely analytical treatment of mechanical questions with the introduction of the conceptions used by Hamilton and Grassmann seems of the greatest importance. It is only when we have reached, as, thanks to the labors of Lagrange, Gauss, and others, we have done, the stage of having set up a purely mathematical model of nature in so far as the motions are mechanical, that it becomes possible to decide with accuracy whether any particular principle of mechanics is of a purely logical nature or whether it depends in the last resort on experience. In fact, if our mathematical model consists of a system of differential equations, and we can deduce by mathematical—that is to say, logical—methods all of the properties which serve as models of actual mechanical occurrences, then we can conclude that any principle of mechanics comes under one of the two following headings: (1) Such principles as have been usually treated rather insufficiently in philosophy, and consist in the enumeration of all the conditions which make possible such a correspondence between nature and logical concepts as is expressed in the mathematical model; (2) those particular conditions which restrict the form of the model. The first set of conditions is usually entirely neglected by men of science; we have examples of such conditions in the principles vaguely alluded to by philosophers as "the principle of the comprehensibility of nature" and "the principle of the uniformity of nature".[30] The second class of principles have been considered more or less unsatisfactorily by men of science. An example is given by the "principle of the conservation of motion of the center of gravity" discovered by Newton. The differential equations of motion are ordinary equations of the second order, but they are not the most general types of such equations; they are restricted in form, and in some important cases allow three integrals to be immediately obtained by addition of the equations of the system so as to form three new equations which are integrable at once. This fact was obviously first observed as a consequence of our observation of dynamical systems, but it is important to deduce this integral, if possible, in a purely logical way from the system of dynamical equations. Only then can we be quite certain that the integral in question does not depend on unanalyzed experiences beyond that involved in the establishment of the equations of motion. The usual statements in textbooks of how this integral may be deduced from the system of dynamical equations completely fail because geometrical and phoronomical appeals are inserted in a manner that may be good for a student but is certainly confusing to a logician. Indeed, Lagrange himself cannot be absolved from making an appeal to intuition which is contrary to the spirit of his analytical ideal of mechanics. After he had deduced the integral of *vis viva* from his well-known equations in generalized coordinates, he merely referred[31] to his previous treatment of it by rectangular coordinates —in which we have to think which directions in space we have to resolve our forces along when we visualize the moving dynamical system, and do not simply hold in our minds the system of equations—and remarked that it would present itself of its own accord in the solution of each problem, provided that the choice of coordinates is properly made.

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[29] Cf. Peano, *Scientia,* Vol. XVIII, 1915, p. 169.

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[30] I have dealt in a very tentative and insufficient manner with these principles in a note to my translation of Mach's *History and Root of the Principle of the Conservation of Energy,* Chicago and London, 1911, pp. 99-101.

It is perfectly true that the integral in question was discovered by phoronomical considerations, and that this fact should be strongly brought out in a psychological study such as a complete history, and that students, as a rule, should be taught such things in a historical way. But we are led by history to another and deeper view of the search for principles: besides the interesting description of how principles were discovered, we have the question as to the logical connection of these principles with one another and with the whole science. It is necessary from this point of view to cut ourselves adrift from irrelevant thoughts as to how somebody or other discovered such and such a principle, and enter into a deeper and purely logical discussion. To such a discussion historical studies are a preliminary, and in the vast majority of cases a necessary preliminary.

For these studies we must pursue a purely analytical method because only by so doing can we be certain of keeping strictly to logical deductions. In pure mathematics the analytical tendency developed especially by Lagrange and Weierstrass can, it seems, be fully justified only by this. When, therefore, we wish to found a logical treatment of mechanics on the new conceptions of Hamilton and Grassmann, we must take care not to introduce these conceptions in a geometrical way, but must use them for the sake of the wonderful simplicity with which they allow us to formulate the principles of mechanics.

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[31] *Méchanique Analitique*, Paris, 1788, p. 240.

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